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Two-Loop Renormalization of Heavy–Light Currents At Order $1/m_Q$ in the Heavy-Quark Expansion

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Abstract

We present exact results, at next-to-leading order in renormalization-group improved perturbation theory, for the Wilson coefficients appearing at order $1/m_Q$ in the heavy-quark expansion of heavy–light current operators. To this end, we complete the calculation of the corresponding two-loop anomalous dimension matrix. Our results are important for determinations of $|V_{ub}|$ using exclusive and inclusive semileptonic B decays. They are also relevant to computations of the decay constant f_B based on a heavy-quark expansion.

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1 Introduction

Hadronic matrix elements of flavor-changing currents are of paramount importance to the phenomenology of weak decays. They appear, e.g., in the theoretical description of semileptonic B decays. These processes give direct access to the elements $|V_{cb}|$ and $|V_{ub}|$ of the quark-mixing matrix, provided the hadronic matrix elements are known from some nonperturbative computation. For inclusive semileptonic decays, the matrix elements can be calculated using a short-distance expansion in the large b-quark mass. Improving the theoretical accuracy in the determinations of $|V_{ub}|$ is currently one of the most important challenges in heavy-flavor theory.

Relevant to the determination of $|V_{ub}|$ are hadronic matrix elements of the weak current $\bar{u}\gamma^\alpha(1-\gamma_5)b$. In general, local current operators composed of a heavy quark Q and a light quark q exhibit an interesting behavior under renormalization at scales below the heavy-quark mass. Large logarithms of the type $\alpha_s \ln(m_Q/\mu)$ arise from the exchange of gluons that are hard with respect to the light quark but soft with respect to the heavy quark. Since such gluons see the heavy quark as a static color source, their effects can be investigated systematically in the framework of the heavy-quark effective theory (HQET), which provides a systematic expansion of matrix elements in inverse powers and logarithms of m_Q [1]. In the HQET, the four-component heavy-quark field is replaced by a velocity-dependent two-component field h_v satisfying $\not{v}\, h_v = h_v$, where v is the four-velocity of the hadron containing the heavy quark. Operators in the effective theory have a different evolution than in usual QCD. For instance, whereas the vector current $\bar{q}\,\gamma^\alpha Q$ is conserved in QCD (i.e., its anomalous dimension vanishes), the corresponding current $\bar{q}\,\gamma^\alpha h_v$ in the HQET has a nontrivial anomalous dimension, which governs its renormalization-group (RG) evolution for scales below m_Q.

As an example, consider the heavy-quark expansion of the vector current (neglecting the light-quark mass m_q) [2, 3]

$$\bar{q}\,\gamma^\alpha Q \cong \sum_{i=1,2} C_i(\mu)\, J_i + \frac{1}{2m_Q} \sum_{j=1}^{10} B_j(\mu)\, Q_j + O(1/m_Q^2)\,, \tag{1}$$

where the symbol $\cong$ is used to indicate that this is a relation that holds after taking matrix elements on both sides. (In a regularization scheme with anticommuting γ_5, the expansion of the axial-vector current is obtained by replacing $\bar{q} \to -\bar{q}\gamma_5$ in the HQET operators.) $J_1 = \bar{q}\,\gamma^\alpha h_v$ and $J_2 = \bar{q}\,v^\alpha h_v$ are the effective operators entering at leading order in the expansion. The operators Q_j appearing at next-to-leading order (NLO) form a basis of dimension-4 operators in the HQET that closes under renormalization. The standard choice of these operators is (using the notation $iD = i\partial + g_s A$ and $(iD)^\dagger = -i\overleftarrow{\partial} + g_s A$)

$$\begin{aligned}
Q_1 &= \bar{q}\,\gamma^\alpha i\not{D} h_v\,, &\qquad Q_4 &= \bar{q}\,(iv\cdot D)^\dagger \gamma^\alpha h_v\,,\\
Q_2 &= \bar{q}\,v^\alpha i\not{D} h_v\,, &\qquad Q_5 &= \bar{q}\,(iv\cdot D)^\dagger v^\alpha h_v\,,\\
Q_3 &= \bar{q}\,iD^\alpha h_v\,, &\qquad Q_6 &= \bar{q}\,(iD^\alpha)^\dagger h_v\,,
\end{aligned} \tag{2}$$

and

$$
\begin{aligned}
Q_{7,8} &= i\int \mathrm{d}^4x\,\mathrm{T}\{J_{1,2}(0), \bar{h}_v(iD)^2 h_v(x)\}\,, \\
Q_{9,10} &= i\int \mathrm{d}^4x\,\mathrm{T}\{J_{1,2}(0), \frac{g_s}{2}\,\bar{h}_v\sigma_{\mu\nu}G^{\mu\nu}h_v(x)\}\,.
\end{aligned} \tag{3}
$$

In addition to the local operators $Q_{1\dots6}$, there appear four bilocal operators $Q_{7\dots10}$ built from time-ordered products of the leading-order currents with an insertion of the "kinetic operator" $\bar{h}_v(iD)^2h_v$ or the "chromo-magnetic operator" $\frac{g_s}{2}\,\bar{h}_v\sigma_{\mu\nu}G^{\mu\nu}h_v$. These operators appear at order $1/m_Q$ in the effective Lagrangian of the HQET. Part of their effect is to take into account the expansion of the external hadron states in terms of states of the effective theory [1]. The Wilson coefficients of these nonlocal operators are given by the products of the coefficients of their component operators, i.e.,

$$
B_{7,8}(\mu) = C_{1,2}(\mu)\,, \qquad B_{9,10}(\mu) = C_{1,2}(\mu)\,C_{\mathrm{mag}}(\mu)\,, \tag{4}
$$

where $C_{\mathrm{mag}}(\mu)$ is the coefficient of the chromo-magnetic operator in the HQET Lagrangian, which has been calculated at NLO in [4]. The kinetic operator is not renormalized [5].

The calculation of the Wilson coefficients $C_i(\mu)$ and $B_j(\mu)$ at NLO in RG-improved perturbation theory requires one-loop matching at the scale $\mu = m_Q$ and two-loop anomalous dimensions in the effective theory. The leading-order coefficients $C_i(\mu)$ have been computed long ago [6, 7]. The calculation at order $1/m_Q$ is, however, much more complicated. The leading logarithmic corrections were obtained in [2]. First steps towards a NLO analysis were made in [3], where the one-loop matching calculations were computed, and general arguments based on the reparameterization invariance [5] of the HQET were used to derive the exact relations (valid to all orders in perturbation theory)

$$
B_1(\mu) = C_1(\mu)\,, \qquad B_2(\mu) = \frac{1}{2}\,B_3(\mu) = C_2(\mu)\,. \tag{5}
$$

Moreover, it was shown that the 10×10 anomalous dimension matrix governing the mixing of the operators Q_j under RG transformations has the texture

$$
\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma_{\mathrm{hl}}} & \boldsymbol{\gamma_A} & 0 \\ 0 & \boldsymbol{\gamma_B} & 0 \\ 0 & \boldsymbol{\gamma_C} & \boldsymbol{\gamma_D} \end{pmatrix}, \tag{6}
$$

where

$$
\begin{aligned}
\boldsymbol{\gamma_{\mathrm{hl}}} &= \mathrm{diag}(\gamma_1,\gamma_1,\gamma_1)\,, \\
\boldsymbol{\gamma_D} &= \mathrm{diag}(\gamma_1,\gamma_1,\gamma_1+\gamma_{\mathrm{mag}},\gamma_1+\gamma_{\mathrm{mag}})\,, \\
(\boldsymbol{\gamma_B})_{ij} &= \gamma_1\,\delta_{ij} + \delta_{i3}\,(\boldsymbol{\gamma_A})_{3j}\,.
\end{aligned} \tag{7}
$$

Here γ_1 and $\gamma_{\rm mag}$ are the anomalous dimensions of the leading-order heavy–light currents $J_{1,2}$ and of the chromo-magnetic operator, respectively. The matrix

$$\boldsymbol{\gamma_A} = \begin{pmatrix} -2(\gamma_2+\gamma_3) & 2\gamma_3 & 2\gamma_2 \\ 0 & 0 & 0 \\ -(\gamma_2+\gamma_3) & \gamma_3 & \gamma_2 \end{pmatrix} \qquad (8)$$

describing the mixing of the local operators $Q_{1\dots3}$ into $Q_{4\dots6}$ has been calculated at two-loop order in [8]. Explicit expression for the two-loop anomalous dimensions will be presented in Section 5.

It was observed in [3] that there are numerically large discrepancies between the one-loop and leading-logarithmic approximations to the Wilson coefficients $B_j(\mu)$, which could only be resolved at NLO in RG-improved perturbation theory. The missing piece in the NLO analysis is the calculation of the anomalous dimension matrix $\boldsymbol{\gamma_C}$ describing the mixing of the bilocal operators $Q_{7\dots10}$ into the local operators $Q_{4\dots6}$. In this paper, we report the calculation of this matrix at two-loop order. It requires the evaluation of two-loop tensor integrals in the HQET, which are infrared (IR) singular when one of the external lines is taken on-shell. A general algorithm for computing such integrals has been developed in [4]. For our analysis we will generalize the above discussion and consider an arbitrary Dirac structure Γ of the current $\bar q\,\Gamma h_v$ in the time-ordered products. At the end we will study the particular case of vector currents in detail.

The reader interested only in the final results of our calculation but not its technical details can continue with Section 5, where we present the final expressions for the Wilson coefficients $B_j(\mu)$ at NLO and briefly discuss some phenomenological applications.

2 Operator mixing

Our task is to calculate, at two-loop order, the mixing of bilocal operators, consisting of time-ordered products of a local current $\bar q\,\Gamma h_v$ with either the kinetic operator or the chromo-magnetic operator, into local dimension-4 current operators with a derivative acting on the light-quark field. Here Γ can be an arbitrary Dirac matrix, depending on the Lorentz structure of the original current that is expanded in terms of HQET operators.

We start by constructing, in the HQET, a basis of the relevant operators that mix under renormalization. Since ultimately our interest is in the matrix elements of these operators between physical hadron states, it is sufficient to consider gauge-invariant operators that do not vanish by the equations of motions. For the case where the bilocal operator contains the kinetic operator, such a basis consists of the two operators

$$\begin{aligned} O_1^{\rm kin} &= \bar q\,(iv\cdot D)^\dagger\Gamma h_v\,, \\ O_T^{\rm kin} &= i\int {\rm d}^4x\,{\rm T}\{\bar q\,\Gamma h_v(0), \bar h_v(iD)^2 h_v(x)\}\,. \end{aligned} \qquad (9)$$

If the bilocal operator contains the magnetic operator, the basis consists of the three operators

$$\begin{aligned}
O_1^{\text{mag}} &= -\frac{1}{4}\,\bar{q}\,(iv\cdot D)^\dagger \sigma_{\mu\nu}\Gamma\,(1+\not{v})\,\sigma^{\mu\nu} h_v\,, \\
O_2^{\text{mag}} &= -\frac{1}{4}\,\bar{q}\,(iD_\nu)^\dagger i\gamma_\mu \not{v}\,\Gamma\,(1+\not{v})\,\sigma^{\mu\nu} h_v\,, \\
O_T^{\text{mag}} &= i\int \mathrm{d}^4x\,\mathrm{T}\{\bar{q}\,\Gamma h_v(0), \frac{g_s}{2}\,\bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v(x)\}\,.
\end{aligned} \tag{10}$$

For each of the two cases, we define a matrix $\boldsymbol{Z}$ of renormalization constants, which absorb the ultraviolet (UV) divergences in the matrix elements of the bare operators, by the relation $O_i = \sum_j Z_{ij}\, O_{j,\text{bare}}$. The matrix $\boldsymbol{\gamma}$ of the anomalous dimensions, which govern the scale dependence of the renormalized operators, is given by

$$\boldsymbol{\gamma} = -\frac{\mathrm{d}\boldsymbol{Z}}{\mathrm{d}\ln\mu}\,\boldsymbol{Z}^{-1}\,. \tag{11}$$

In a minimal subtraction scheme, the renormalization constants are defined to remove the $1/\epsilon$ poles arising in the calculation of the bare Green functions with insertions of the operators O_i in dimensional regularization, i.e. in $d = 4-2\epsilon$ space–time dimensions. Hence,

$$\boldsymbol{Z} = \boldsymbol{1} + \sum_{k=0}^{\infty} \frac{1}{\epsilon^k}\,\boldsymbol{Z}^{(k)}\,. \tag{12}$$

The requirement that the anomalous dimension matrix be finite in the limit $\epsilon \to 0$ implies the relations [9]

$$\boldsymbol{\gamma} = 2\alpha_s \frac{\partial \boldsymbol{Z}^{(1)}}{\partial\alpha_s}\,, \qquad \alpha_s \frac{\partial \boldsymbol{Z}^{(2)}}{\partial\alpha_s} = \alpha_s \frac{\partial \boldsymbol{Z}^{(1)}}{\partial\alpha_s}\left(\boldsymbol{Z}^{(1)} + \frac{\beta(\alpha_s)}{\alpha_s}\right)\,, \tag{13}$$

where $\beta(\alpha_s) = \mathrm{d}\alpha_s/\mathrm{d}\ln\mu^2$ is the β function. The first equation shows that the anomalous dimension matrix can be obtained from the coefficient of the $1/\epsilon$ pole in $\boldsymbol{Z}$, whereas the second one implies a nontrivial constraint on the coefficient of the $1/\epsilon^2$ pole, arising at two-loop and higher order.

In our case, the structure of the two $\boldsymbol{Z}$ matrices is

$$\boldsymbol{Z}^{\text{kin}} = \begin{pmatrix} Z_1 & 0 \\ Z_{T1}^{\text{kin}} & Z_1 \end{pmatrix}\,, \qquad \boldsymbol{Z}^{\text{mag}} = \begin{pmatrix} Z_1 & 0 & 0 \\ Z_4 & Z_1+Z_2 & 0 \\ Z_{T1}^{\text{mag}} & Z_{T2}^{\text{mag}} & Z_1 Z_{\text{mag}} \end{pmatrix}\,, \tag{14}$$

where the renormalization constants Z_1, Z_2 and Z_4 determine the mixing of local, dimension-4 heavy–light currents and have been calculated at two-loop order in [8], whereas Z_{mag} is the renormalization constant of the chromo-magnetic operator, which has been computed at two-loop order in [4]. Here we will calculate the remaining entries Z_{Ti} with the same accuracy.

Previous authors have calculated the mixing of the bilocal operators into local operators at the one-loop order, finding [2, 3]

$$Z_{T1}^{\rm kin} = -\frac{C_F\alpha_s}{\pi\epsilon} + O(\alpha_s^2)\,, \qquad Z_{T1}^{\rm mag} = Z_{T2}^{\rm mag} = -\frac{C_F\alpha_s}{4\pi\epsilon} + O(\alpha_s^2)\,. \tag{15}$$

Using the known expression for the other Z-factors, this information can be used to predict the $1/\epsilon^2$ poles in the two-loop coefficients of the renormalization constants. From the second relation in (13), we obtain

$$\begin{aligned}
Z_{T1}^{\rm kin\,(2)} &= \left(\frac{\alpha_s}{4\pi}\right)^2 \left[6C_F^2 + \frac{22}{3}\,C_F C_A - \frac{8}{3}\,C_F T_F\,n_f\right] + O(\alpha_s^3)\,, \\
Z_{T1}^{\rm mag\,(2)} &= \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{7}{4}\,C_F^2 + \frac{4}{3}\,C_F C_A - \frac{2}{3}\,C_F T_F\,n_f\right] + O(\alpha_s^3)\,, \\
Z_{T2}^{\rm mag\,(2)} &= \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{3}{4}\,C_F^2 + \frac{4}{3}\,C_F C_A - \frac{2}{3}\,C_F T_F\,n_f\right] + O(\alpha_s^3)\,.
\end{aligned} \tag{16}$$

Here $C_A = N$, $C_F = \frac{1}{2}(N^2-1)/N$ and $T_F = \frac{1}{2}$ are the color factors for an $SU(N)$ gauge group, and n_f is the number of light-quark flavors. These relations will provide a check on our two-loop results.

3 Two-loop calculation

To obtain the renormalization constants Z_{Ti} at order α_s^2, we calculate the insertions of the bilocal operators $O_T^{\rm kin}$ and $O_T^{\rm mag}$ into the amputated Green function with a heavy and a light quark to two-loop order. The relevant diagrams are shown in Figure 1. We use dimensional regularization in $d = 4 - 2\epsilon$ space–time dimensions and work in the $\overline{\rm MS}$ subtraction scheme. The relation between the renormalized and the bare, regularized amputated Green functions is

$$\Gamma_{{\rm ren},i}(v,p,\ldots) = (Z_h Z_q)^{1/2} Z_{ij}\,\Gamma_{{\rm reg},j}(v,p,\ldots;\epsilon) = \text{finite}. \tag{17}$$

To compute the mixing of the bilocal operators into the local operators with a derivative acting on the light-quark field, we only need to keep terms linear in the momentum p of the light quark. Because the pole parts are polynomial in the external momenta, we can first take a derivative with respect to p and then set $p = 0$, so that all integrals are of propagator type and depend on the single variable $\omega = v\cdot k$. However, this method fails for some of the diagrams, for which setting $p = 0$ after differentiation leads to infrared (IR) divergences. In these cases, we apply a variant of the so-called R^* operation [4, 10], which compensates these IR poles by a recursive construction of counterterms for the IR-divergent subgraphs.

Consider, as an example, the first diagram in Figure 1. Its contribution is proportional to the integral

$$D_1 = \int {\rm d}^d s\,{\rm d}^d t\,\frac{(s-t)_\alpha\,(t+p)_\beta\,(s+p)_\gamma}{t^2(t+p)^2(s+p)^2(s-t)^2(v\cdot s+\omega)(v\cdot t+\omega)}\,. \tag{18}$$

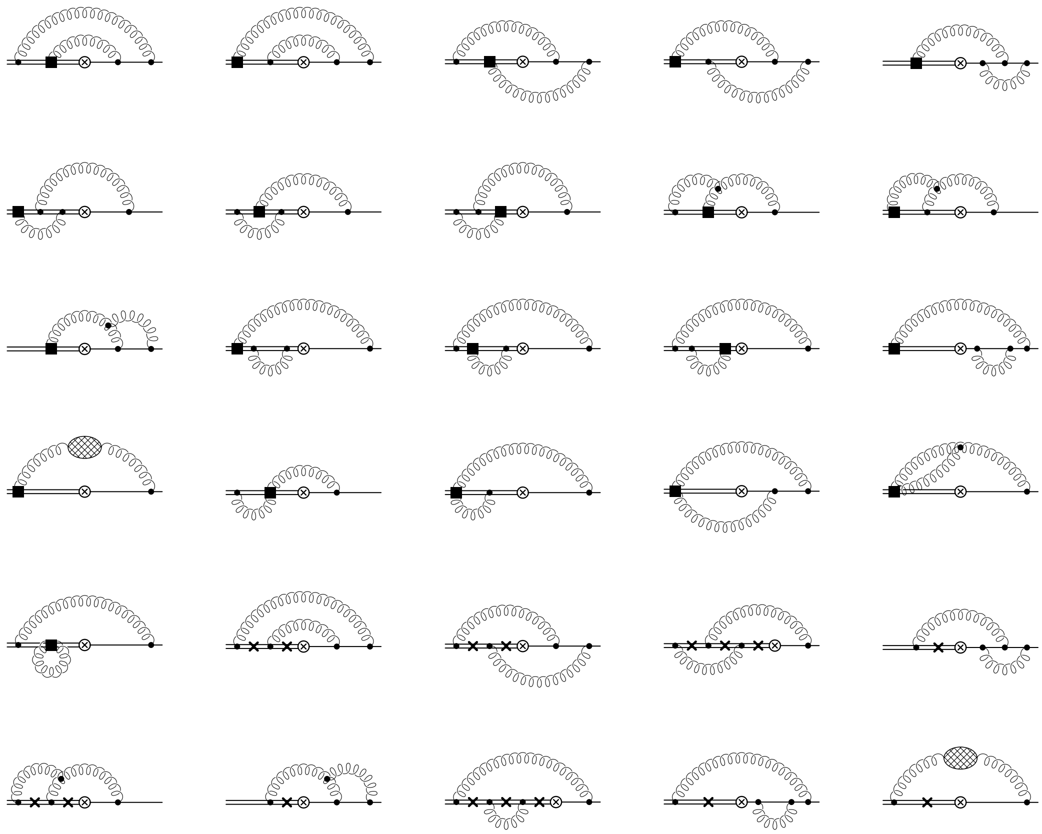

Figure 1: Two-loop diagrams contributing to the renormalization constants Z_{Ti}. The heavy quark is represented by a double line. The crossed circles represent the leading-order current, while the black squares represent an insertion of the kinetic or the chromo-magnetic operator. The shaded blobs represent one-loop insertions of the gluon self-energy. The last nine diagrams contribute only to insertions of the kinetic operator. In these graphs, the crosses on heavy-quark lines with momentum p mark the different positions where we insert ip^2.

When linearizing this expression in p, we encounter an IR divergence from the region $t \to 0$, which can be removed by adding and subtracting the IR subtraction term

$$D_1^{\rm IR} = \int \mathrm{d}^d s \, \frac{s_\alpha \, s_\gamma}{(s^2)^2 (v\cdot s + \omega)\, \omega} \int \mathrm{d}^d t \, \frac{(t+p)_\beta}{t^2 (t+p)^2} \,. \tag{19}$$

The difference $D_1 - D_1^{\rm IR}$ can be evaluated using a naive linearization in p, since the behavior of the IR subtraction term for $t \to 0$ is the same as that of the original integral. However, because the expansion of $D_1^{\rm IR}$ involves tadpole integrals that vanish in dimensional regularization, the difference $(D_1 - D_1^{\rm IR})_{\rm linearized}$ coincides with the naive linearization of the original expression for D_1. The contribution $D_1^{\rm IR}$, which is necessary to

subtract the IR subdivergence of the original diagram, factorizes into an IR counterterm

$$\int \mathrm{d}^d t\, \frac{\gamma^\beta (t+p)_\beta}{t^2 (t+p)^2} = i\pi^{d/2}\, \not{p}\, (-p^2)^{-\epsilon}\, \frac{\Gamma(d/2)\,\Gamma(d/2-1)\,\Gamma(2-d/2)}{\Gamma(d-1)}\,. \tag{20}$$

and the original diagram with the lines of the IR-sensitive subgraph removed, and with t and p set to zero.

The remaining two-loop tensor integrals are of the general form

$$\begin{aligned}\int \mathrm{d}^d s\, \mathrm{d}^d t\, \left(\frac{\omega}{v\cdot s+\omega}\right)^{\alpha_1} \left(\frac{\omega}{v\cdot t+\omega}\right)^{\alpha_2} \frac{s_{\mu_1}\dots s_{\mu_n} t^{\nu_1}\dots t^{\nu_m}}{(-s^2)^{\alpha_3}(-t^2)^{\alpha_4}[-(s-t)^2]^{\alpha_5}} \\ \equiv -\pi^d (-2\omega)^{2(d-\alpha_3-\alpha_4-\alpha_5)+n+m}\, I^{\nu_1\dots\nu_m}_{\mu_1\dots\mu_n}(v;\{\alpha_i\})\,.\end{aligned} \tag{21}$$

Using the method of integration by parts [7, 11], one obtains the recurrence relation [4]

$$\begin{aligned}\left[(d-\alpha_1-\alpha_3-2\alpha_5+n) + \alpha_3\, \mathbf{3}^+(\mathbf{4}^- - \mathbf{5}^-) + \alpha_1\, \mathbf{1}^+\mathbf{2}^-\right]\, I^{\nu_1\dots\nu_m}_{\mu_1\dots\mu_n}(v;\{\alpha_i\}) \\ = \sum_{j=1}^n I^{\mu_j\nu_1\dots\nu_m}_{\mu_1\dots[\mu_j]\dots\mu_n}(v;\{\alpha_i\})\,,\end{aligned} \tag{22}$$

which can be used to express any two-loop integral in terms of degenerate integrals, which have $\alpha_2 = 0$, $\alpha_4 = 0$ or $\alpha_5 = 0$. Here $\mathbf{1}^+$ is an operator raising the index α_1 by one unit etc., and $[\mu_j]$ means that this index is omitted. The degenerate integrals can be related in a straightforward way to products of one-loop tensor integrals [4].

Using this technique, we have calculated the pole parts of the two-loop diagrams in the 't Hooft–Feynman gauge. The result is a linear combination of terms with the Dirac structures corresponding to the local operators $O_1^{\rm kin}$ and $O_{1,2}^{\rm mag}$ defined in (9) and (10). The results are summarized in the first two columns of Table 1. In the $\overline{\rm MS}$ scheme, the renormalization scale μ is introduced by the replacement of the bare coupling constant with the renormalized one through the relation $g_s^{\rm bare} = \tilde\mu^\epsilon Z_g\, g_s$ with $\tilde\mu = \mu\, e^{\gamma_E/2}(4\pi)^{-1/2}$.

The two-loop diagrams in Figure 1 contain subdivergences, which must be subtracted by UV counterterms. In addition to the standard one-loop counterterms for the quark and gluon propagators and vertices, local operator counterterms are required. To find these, we have calculated at one-loop order all insertions of the operators $O_T^{\rm kin}$ and $O_T^{\rm mag}$ into the amputated Green functions with a non-negative degree of divergence. In our case, those are the two- and three-point functions with field content $\bar q h_v$, $\bar q A h_v$, $\bar h_v h_v$, and $\bar h_v A h_v$. We find that in the 't Hooft–Feynman gauge the UV divergences of these functions are removed by the counterterms

$$\begin{aligned}\mathcal{L}^{\rm kin}_{\rm c.t.} = &-\frac{C_F\alpha_s}{4\pi\epsilon}\Big[4O^{\rm kin} - O_T^{\rm kin} - 2\bar q\,\Gamma\, iv\cdot D\, h_v - \bar q\,(i\not{D})^\dagger \not{v}\,\Gamma h_v \\ &\qquad + i\int \mathrm{d}^4x\, \mathrm{T}\{\bar q\,\Gamma h_v(0), \bar h_v (iv\cdot D)^2 h_v(x)\}\Big] \\ &-\frac{C_A\alpha_s}{4\pi\epsilon}\Big[-\frac{3}{2}\,\bar q\,\Gamma\, v\cdot A\, h_v - \frac{3}{2}\, i\int \mathrm{d}^4x\, \mathrm{T}\{\bar q\,\Gamma h_v(0), \bar h_v\, \{iv\cdot D, v\cdot A\} h_v(x)\} \\ &\qquad + i\int \mathrm{d}^4x\, \mathrm{T}\{\bar q\,\Gamma h_v(0), \bar h_v\, \{iD_\mu, A^\mu\} h_v(x)\}\Big]\,,\end{aligned} \tag{23}$$

Table 1: Two-loop and counterterm contributions in units of $(\alpha_s/4\pi)^2$

Structure	Color	$\left(\frac{-2\omega}{\mu}\right)^{-4\epsilon}$	$\left(\frac{-2\omega}{\mu}\right)^{-2\epsilon}\left(\frac{-p^2}{\mu^2}\right)^{-\epsilon}$	$\left(\frac{-2\omega}{\mu}\right)^{-2\epsilon}$	$\left(\frac{-p^2}{\mu^2}\right)^{-\epsilon}$
$O_1^{\rm kin}$	C_F^2	$\frac{4}{\epsilon^2}+\left(-\frac{4}{3}+\frac{8\pi^2}{9}\right)\frac{1}{\epsilon}$	$-\frac{4}{\epsilon}$	$-\frac{8}{\epsilon^2}+\frac{8}{\epsilon}$	0
	$C_F C_A$	$\frac{53}{6\epsilon^2}+\left(-\frac{287}{18}-\frac{4\pi^2}{3}\right)\frac{1}{\epsilon}$	$-\frac{3}{2\epsilon^2}-\frac{1}{\epsilon}$	$-\frac{97}{6\epsilon^2}+\frac{185}{6\epsilon}$	$\frac{3}{2\epsilon^2}+\frac{3}{\epsilon}$
	$C_F T_F\, n_f$	$-\frac{8}{3\epsilon^2}+\frac{56}{9\epsilon}$	0	$\frac{16}{3\epsilon^2}-\frac{32}{3\epsilon}$	0
$O_1^{\rm mag}$	C_F^2	$\frac{9}{4\epsilon^2}+\left(\frac{13}{3}+\frac{10\pi^2}{9}\right)\frac{1}{\epsilon}$	$-\frac{1}{\epsilon^2}-\frac{6}{\epsilon}$	$-\frac{7}{2\epsilon^2}+\frac{1}{\epsilon}$	$\frac{1}{\epsilon^2}+\frac{2}{\epsilon}$
	$C_F C_A$	$\frac{4}{3\epsilon^2}+\left(\frac{25}{18}-\frac{5\pi^2}{18}\right)\frac{1}{\epsilon}$	0	$-\frac{8}{3\epsilon^2}$	0
	$C_F T_F\, n_f$	$-\frac{2}{3\epsilon^2}-\frac{1}{9\epsilon}$	0	$\frac{4}{3\epsilon^2}$	0
$O_2^{\rm mag}$	C_F^2	$\frac{5}{4\epsilon^2}+\left(-\frac{11}{6}+\frac{2\pi^2}{9}\right)\frac{1}{\epsilon}$	$-\frac{1}{\epsilon^2}-\frac{2}{\epsilon}$	$-\frac{3}{2\epsilon^2}$	$\frac{1}{\epsilon^2}+\frac{2}{\epsilon}$
	$C_F C_A$	$\frac{4}{3\epsilon^2}+\left(\frac{55}{18}-\frac{\pi^2}{18}\right)\frac{1}{\epsilon}$	0	$-\frac{8}{3\epsilon^2}$	0
	$C_F T_F\, n_f$	$-\frac{2}{3\epsilon^2}-\frac{1}{9\epsilon}$	0	$\frac{4}{3\epsilon^2}$	0

and

$$\mathcal{L}_{\rm c.t.}^{\rm mag} = -\frac{C_F\alpha_s}{4\pi\epsilon}\Big[O_1^{\rm mag}+O_2^{\rm mag}-O_T^{\rm mag}+\bar q\sigma_{\mu\nu}\Gamma P_+\,\sigma^{\mu\nu} iv\!\cdot\! D\,h_v\Big]\,. \tag{24}$$

Since the two-loop calculation was performed off-shell, some gauge-dependent operators must be included in addition to the operators O_i and O_T, as well as operators that vanish by the equations of motion. (The local operators vanishing by the equations of motion actually yield no contribution and could be dropped; however, one must keep these operators when they appear inside time-ordered products.) The results for the counterterm contributions are summarized in the last two columns of Table 1. When these contributions are added to the result for the sum of the two-loop diagrams, all nonlocal $1/\epsilon$ divergences proportional to $\ln(-2\omega/\mu)$ and $\ln(-p^2/\mu^2)$ cancel. This is a check of our calculation.

It follows from (17) that the sum of the two-loop diagrams plus their counterterms determines the two-loop coefficients of the products $-(Z_h Z_q)^{1/2} Z_{Ti}$. To obtain the results for the renormalization constants Z_{Ti} at order α_s^2, we have to account for one-loop operator mixing and wave-function renormalization of the external quark fields. Our final expressions are

$$\begin{aligned} Z_{T1}^{\rm kin} &= \frac{C_F\alpha_s}{\pi\epsilon}\bigg\{-4+\frac{\alpha_s}{4\pi}\bigg[C_F\left(\frac{6}{\epsilon}-\frac{8}{3}-\frac{8\pi^2}{9}\right)+C_A\left(\frac{22}{3\epsilon}-\frac{152}{9}+\frac{4\pi^2}{3}\right) \\ &\qquad + T_F\, n_f\left(-\frac{8}{3\epsilon}+\frac{40}{9}\right)\bigg]\bigg\}\,, \end{aligned}$$

$$
\begin{aligned}
Z_{T1}^{\rm mag} &= \frac{C_F\alpha_s}{4\pi\epsilon}\bigg\{-1+\frac{\alpha_s}{4\pi}\bigg[C_F\left(\frac{7}{4\epsilon}-\frac{4}{3}-\frac{10\pi^2}{9}\right)+C_A\left(\frac{4}{3\epsilon}-\frac{25}{8}+\frac{5\pi^2}{18}\right)\\
&\qquad +T_F\,n_f\left(-\frac{2}{3\epsilon}+\frac{1}{9}\right)\bigg]\bigg\}\,,\\
Z_{T2}^{\rm mag} &= \frac{C_F\alpha_s}{4\pi\epsilon}\bigg\{-1+\frac{\alpha_s}{4\pi}\bigg[C_F\left(\frac{3}{4\epsilon}+\frac{11}{6}-\frac{2\pi^2}{9}\right)+C_A\left(\frac{4}{3\epsilon}-\frac{55}{18}+\frac{\pi^2}{18}\right)\\
&\qquad +T_F\,n_f\left(-\frac{2}{3\epsilon}+\frac{1}{9}\right)\bigg]\bigg\}\,.
\end{aligned}
\tag{25}
$$

They are the main result of this work. Note that the $1/\epsilon^2$ poles in these expressions agree with (16). This a nontrivial check of our calculation.

4 Nontrivial basis transformations

The results derived in this work are sufficient to calculate, at the two-loop order, the operator mixing of bilocal operators into local dimension-4 operators for any choice of the Dirac structure Γ. However, care must be taken when using the results in (25) for a particular choice of Γ, if instead of the operators in (9) and (10) another set of basis operators is employed. Whereas the transformation between one operator basis and another is trivial at the one-loop order, it can be subtle at NLO, because in dimensional regularization the relations between the operators of different bases may depend on ϵ.

Consider the general case where the operators $\{O_i\}$ are expressed in terms of some other operators $\{Q_j\}$ by a linear transformation of the form $O_i=\sum_j R_{ij}(\epsilon)\,Q_j$. Depending on the choice of Γ, some of the operators O_i may not be independent, so the new set may contain fewer operators than the original one. We define a left-inverse matrix $\boldsymbol{L}(\epsilon)$ such that $\boldsymbol{L}(\epsilon)\boldsymbol{R}(\epsilon)=\boldsymbol{1}$. It then follows that the matrix $\tilde{\boldsymbol{Z}}$ of renormalization constants in the new basis (denoted by a tilde) must be such that[1]

$$
\boldsymbol{L}\boldsymbol{Z}\boldsymbol{R}\,\tilde{\boldsymbol{Z}}^{-1}\Big|_{\rm poles}=0\,.
\tag{26}
$$

Next, we expand the transformation matrices as $\boldsymbol{R}(\epsilon)=\sum_n \boldsymbol{R_n}(\epsilon)\,\epsilon^n$ and $\boldsymbol{L}(\epsilon)=\sum_n \boldsymbol{L_n}(\epsilon)\,\epsilon^n$ with $n\ge 0$, and extract from this result the coefficient of the $1/\epsilon$ pole in the matrix $\tilde{\boldsymbol{Z}}$, which determines the anomalous dimension matrix in the new basis. We find

$$
\begin{aligned}
\tilde{\boldsymbol{Z}}^{(1)} &= \boldsymbol{L_0}\boldsymbol{Z}^{(1)}\boldsymbol{R_0}+\Big[\boldsymbol{L_1}\boldsymbol{Z}^{(2)}\boldsymbol{R_0}+\boldsymbol{L_0}\boldsymbol{Z}^{(2)}\boldsymbol{R_1}\Big]\\
&\qquad -\Big[\boldsymbol{L_1}\boldsymbol{Z}^{(1)}\boldsymbol{R_0}+\boldsymbol{L_0}\boldsymbol{Z}^{(1)}\boldsymbol{R_1}\Big]\boldsymbol{L_0}\boldsymbol{Z}^{(1)}\boldsymbol{R_0}+\dots\,,
\end{aligned}
\tag{27}
$$

where the dots represent terms that do not contribute at two-loop order.

[1] The corresponding relation (19) of [8] is incorrect; however, this did not affect the results for the anomalous dimensions obtained in that paper.

Armed with this general result, we now discuss the case of the vector current considered in the Introduction. Then the relevant Dirac structures are $\Gamma = \gamma^\alpha$ and $\Gamma = v^\alpha$, and the operators O_i are conveniently expressed in terms of the operators $Q_{4\dots6}$ defined in (2). For $\Gamma = \gamma^\alpha$ we have $O_1^{\rm kin} = Q_4$, $O_T^{\rm kin} = Q_7$, $O_T^{\rm mag} = Q_9$, and

$$\begin{aligned} O_1^{\rm mag} &= (1-\epsilon)(1+2\epsilon)Q_4 - 4(1-\epsilon)Q_5 \,, \\ O_2^{\rm mag} &= \epsilon\, Q_4 - 2(1-\epsilon)Q_5 + (1-2\epsilon)Q_6 \,, \end{aligned} \tag{28}$$

whereas for $\Gamma = v^\alpha$ we find $O_1^{\rm kin} = Q_5$, $O_T^{\rm kin} = Q_8$, $O_T^{\rm mag} = Q_{10}$, and

$$\begin{aligned} O_1^{\rm mag} &= -(1-\epsilon)(3-2\epsilon)Q_5 \,, \\ O_2^{\rm mag} &= -(1-\epsilon)Q_5 \,. \end{aligned} \tag{29}$$

Hence only for the case of insertions of the magnetic operator there appear nontrivial basis transformations.

We thus consider the basis transformation from the six operators consisting of two sets of $\{O_1^{\rm mag}, O_2^{\rm mag}, O_T^{\rm mag}\}$ evaluated with $\Gamma = \gamma^\alpha$ and $\Gamma = v^\alpha$ to the new basis spanned by the operators $\{Q_4, Q_5, Q_6, Q_9, Q_{10}\}$. The corresponding transformation matrices $\boldsymbol{R}(\epsilon)$ and $\boldsymbol{L}(\epsilon)$ can easily be obtained from (28) and (29). It is then straightforward to compute the matrix $\boldsymbol{Z_C}$ corresponding to the operator mixing of $Q_{7\dots10}$ into $Q_{4\dots6}$, and then using the first relation in (13) to calculate the two-loop anomalous dimension matrix $\boldsymbol{\gamma_C}$. We find that

$$\boldsymbol{\gamma_C} = \begin{pmatrix} \gamma_1^{\rm kin} & 0 & 0 \\ 0 & \gamma_1^{\rm kin} & 0 \\ \gamma_1^{\rm mag} & \gamma_2^{\rm mag} & \gamma_3^{\rm mag} \\ 0 & \gamma_1^{\rm mag} + \gamma_2^{\rm mag} + \gamma_3^{\rm mag} & 0 \end{pmatrix} , \tag{30}$$

where

$$\begin{aligned} \gamma_1^{\rm kin} &= \frac{C_F\alpha_s}{4\pi}\bigg\{ -8 + \frac{\alpha_s}{4\pi}\bigg[C_F\left(-\frac{32}{3} - \frac{32\pi^2}{9}\right) + C_A\left(-\frac{608}{9} + \frac{16\pi^2}{3}\right) + \frac{160}{9}\,T_F\,n_f\bigg]\bigg\} , \\ \gamma_1^{\rm mag} &= \frac{C_F\alpha_s}{4\pi}\bigg\{ -2 + \frac{\alpha_s}{4\pi}\bigg[C_F\left(-\frac{10}{3} - \frac{40\pi^2}{9}\right) + C_A\left(\frac{46}{9} + \frac{10\pi^2}{9}\right) - \frac{44}{9}\,T_F\,n_f\bigg]\bigg\} , \\ \gamma_2^{\rm mag} &= \frac{C_F\alpha_s}{4\pi}\bigg\{ \;\; 12 + \frac{\alpha_s}{4\pi}\bigg[C_F\left(\frac{38}{3} + \frac{176\pi^2}{9}\right) + C_A\left(\frac{236}{3} - \frac{44\pi^2}{9}\right) - \frac{56}{3}\,T_F\,n_f\bigg]\bigg\} , \\ \gamma_3^{\rm mag} &= \frac{C_F\alpha_s}{4\pi}\bigg\{ -2 + \frac{\alpha_s}{4\pi}\bigg[C_F\left(\frac{4}{3} - \frac{8\pi^2}{9}\right) + C_A\left(-\frac{206}{9} + \frac{2\pi^2}{9}\right) + \frac{52}{9}\,T_F\,n_f\bigg]\bigg\} . \end{aligned} \tag{31}$$

This completes the calculation of the anomalous dimension matrix (6) at two-loop order.

5 Wilson coefficients for the vector current

The Wilson coefficients $B_j(\mu)$ in the heavy-quark expansion of the vector current in (1) obey the renormalization-group equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}\ln\mu} - \boldsymbol{\gamma}^T \right) \vec{B}(\mu) = 0\,, \tag{32}$$

where the vector $\vec{B}(\mu)$ contains the ten coefficients $B_j(\mu)$. The anomalous dimension matrix $\boldsymbol{\gamma}$ has been given in (6), (7), (8), and (30). Besides the entries of the submatrix $\boldsymbol{\gamma_C}$ computed in this paper and shown in (31), we need the two-loop anomalous dimension

$$\gamma^{\rm mag} = \frac{C_A\alpha_s}{4\pi} \left\{ 2 + \frac{\alpha_s}{4\pi} \left[\frac{68}{9}\,C_A - \frac{52}{9}\,T_F\,n_f \right] \right\} \tag{33}$$

of the chromo-magnetic operator calculated in [4], as well as the two-loop anomalous dimensions of local dimension-4 currents,

$$\begin{aligned}
\gamma_1 &= \frac{C_F\alpha_s}{4\pi} \left\{ -3 + \frac{\alpha_s}{4\pi} \left[C_F \left(\frac52 - \frac{8\pi^2}{3} \right) + C_A \left(-\frac{49}{6} + \frac{2\pi^2}{3} \right) + \frac{10}{3}\,T_F\,n_f \right] \right\} , \\
\gamma_2 &= \frac{C_F\alpha_s}{4\pi} \left\{ 3 + \frac{\alpha_s}{4\pi} \left[C_F \left(-5 + \frac{4\pi^2}{3} \right) + C_A \left(\frac{41}{3} - \frac{\pi^2}{3} \right) - \frac{10}{3}\,T_F\,n_f \right] \right\} , \\
\gamma_3 &= \frac{C_F\alpha_s}{4\pi} \left\{ -2 + \frac{\alpha_s}{4\pi} \left[C_F \left(\frac{13}{3} - \frac{8\pi^2}{9} \right) + C_A \left(-\frac{104}{9} + \frac{2\pi^2}{9} \right) + \frac{28}{9}\,T_F\,n_f \right] \right\} ,
\end{aligned} \tag{34}$$

which were computed in [8]. The coefficient γ_1 coincides with the universal hybrid anomalous dimension of the leading-order heavy–light currents first obtained in [6, 7]. In addition to the anomalous dimensions, we need the one-loop matching conditions for the Wilson coefficients at the scale $\mu = m_Q$. In the $\overline{\rm MS}$ subtraction scheme, they are [3]

$$\begin{aligned}
C_1(m_Q) &= B_1(m_Q) = 1 - \frac{C_F\alpha_s}{\pi}\,, \\
C_2(m_Q) &= B_2(m_Q) = \frac12\,B_3(m_Q) = \frac{C_F\alpha_s}{2\pi}\,, \\
-B_4(m_Q) &= \frac13\,B_5(m_Q) = B_6(m_Q) = \frac{C_F\alpha_s}{\pi}\,, \\
C_{\rm mag}(m_Q) &= 1 + (C_A + C_F)\,\frac{\alpha_s}{2\pi}\,.
\end{aligned} \tag{35}$$

The solution of the renormalization-group equation at next-to-leading order is standard and described in detail in [3]. A subtlety complicating the solution is that the one-loop anomalous dimension matrix cannot be diagonalized. We circumvent this problem by adding a small diagonal contribution $\eta\,\mathbf{1}_{3\times3}$ to the matrix $\boldsymbol{\gamma_B}$ in (6), taking the limit $\eta\to 0$ at the end of the calculation.

We quote results for $N = 3$ colors and $n_f = 4$ light quark flavors. We first reproduce the known NLO results

$$\begin{aligned} C_1(\mu) &= B_1(\mu) = x^{6/25} \left[1 - 6.243\, \frac{\alpha_s(m_Q)}{4\pi} + (0.910 - 2\Delta_{\rm RS})\, \frac{\alpha_s(\mu)}{4\pi} \right] , \\ C_2(\mu) &= B_2(\mu) = \frac{1}{2}\, B_3(\mu) = \frac{8}{3}\, x^{6/25}\, \frac{\alpha_s(m_Q)}{4\pi} , \\ C_{\rm mag}(\mu) &= x^{-9/25} \left[1 + 8.449\, \frac{\alpha_s(m_Q)}{4\pi} + (0.218 + 3\Delta_{\rm RS})\, \frac{\alpha_s(\mu)}{4\pi} \right] , \end{aligned} \tag{36}$$

where $x = \alpha_s(\mu)/\alpha_s(m_Q)$. According to (4), products of these coefficients determine the coefficients $B_{7\dots10}$. In addition, we obtain the new expressions

$$\begin{aligned} B_4(\mu) &= x^{6/25} \left[\frac{34}{27} - 9.127\, \frac{\alpha_s(m_Q)}{4\pi} + \left(2.820 - \frac{212}{27}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad + x^{6/25} \ln x \left[\frac{16}{25} - 3.996\, \frac{\alpha_s(m_Q)}{4\pi} + \left(0.582 - \frac{32}{25}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad + x^{-3/25} \left[-\frac{4}{27} - 0.327\, \frac{\alpha_s(m_Q)}{4\pi} + \left(2.013 - \frac{4}{27}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad - \frac{10}{9} + 0.280\, \frac{\alpha_s(m_Q)}{4\pi} - 0.993\, \frac{\alpha_s(\mu)}{4\pi} , \\ B_5(\mu) &= x^{6/25} \left[-\frac{28}{27} + (32.097 + 1.707 \ln x)\, \frac{\alpha_s(m_Q)}{4\pi} - \left(0.944 - \frac{56}{27}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad + x^{-3/25} \left[\frac{88}{27} + 11.930\, \frac{\alpha_s(m_Q)}{4\pi} - \left(30.101 - \frac{88}{27}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad - \frac{20}{9} + 0.560\, \frac{\alpha_s(m_Q)}{4\pi} + 2.458\, \frac{\alpha_s(\mu)}{4\pi} , \\ B_6(\mu) &= x^{6/25} \left[-2 + 7.153\, \frac{\alpha_s(m_Q)}{4\pi} - (1.820 - 4\Delta_{\rm RS})\, \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad + x^{-3/25} \left[-\frac{4}{3} - 2.941\, \frac{\alpha_s(m_Q)}{4\pi} + \left(5.246 - \frac{4}{3}\, \Delta_{\rm RS} \right) \frac{\alpha_s(\mu)}{4\pi} \right] \\ &\quad + \frac{10}{3} - 0.841\, \frac{\alpha_s(m_Q)}{4\pi} - 1.464\, \frac{\alpha_s(\mu)}{4\pi} . \end{aligned} \tag{37}$$

All terms proportional to the coupling $\alpha_s(m_Q)$ in these expressions, as well as the leading-logarithmic rescaling factors, are independent of the renormalization scheme. The terms proportional to $\alpha_s(\mu)$ are scheme dependent, however. Here we consider a

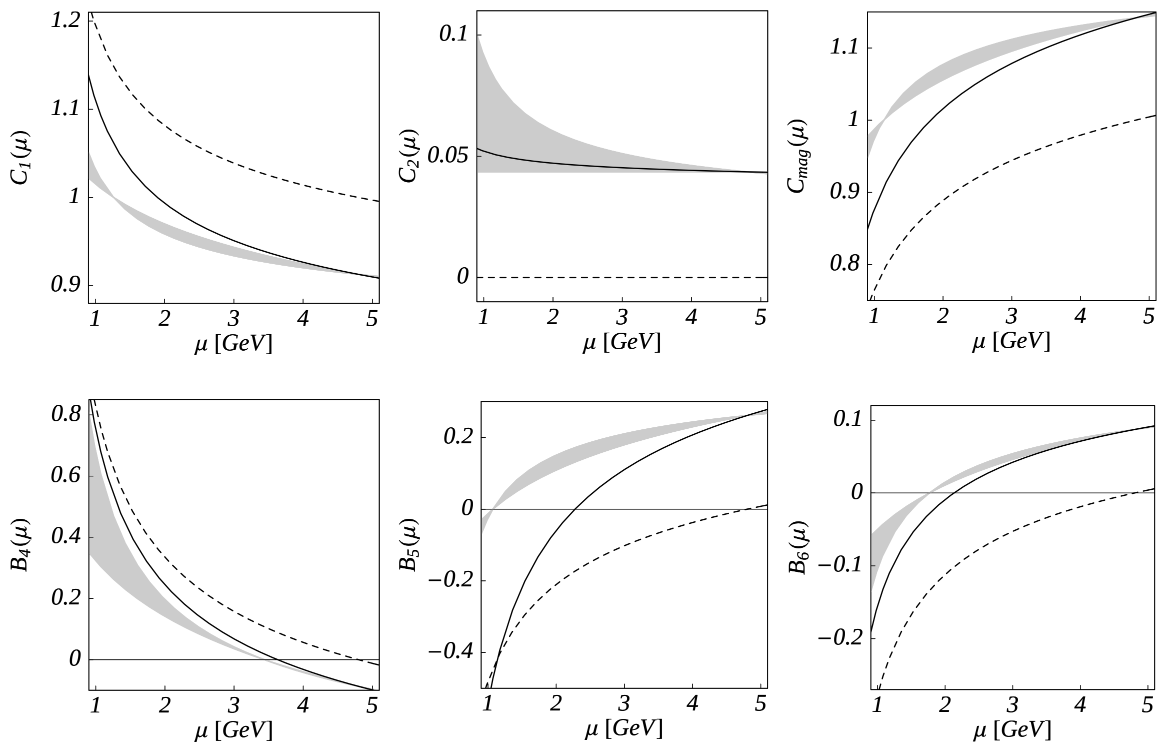

Figure 2: Different perturbative approximations for the Wilson coefficients in the $\overline{\rm MS}$ scheme: next-to-leading order (solid), leading-logarithmic approximation (dashed), and one-loop order (band).

class of schemes parameterized by the quantity

$$\Delta_{\rm RS} = \ln \frac{\mu^2_{\overline{\rm MS}}}{\mu^2_{\rm RS}} \,. \tag{38}$$

These include all "minimal-subtraction-like" schemes, which are related to each other by a change in the renormalization scale. For instance, we have $\Delta_{\rm MS} = -\gamma_E + \ln 4\pi$ and $\Delta_{\overline{\rm MS}} = 0$ by definition, i.e., the above results evaluated with $\Delta_{\rm RS} = 0$ refer to the $\overline{\rm MS}$ scheme. It follows from (32) that at NLO the scheme-dependent terms in the Wilson coefficients are given by

$$\Delta\vec{B}(\mu) = \frac{\Delta_{\rm RS}}{2}\,\boldsymbol{\gamma}^T\,\vec{B}(\mu)\,. \tag{39}$$

In a complete NLO calculation, these terms combine with scheme-dependent terms in the hadronic matrix elements of the HQET operators to give a renormalization-group invariant answer.

It is evident from the magnitude of some of the coefficients in (37) that the next-to-leading corrections are, in some cases, quite large. To illustrate this point, we compare our NLO results with the naive one-loop expressions for the Wilson coefficients calculated

in [3], and with the results obtained in leading-logarithmic approximation [2], where all terms proportional to $\alpha_s/4\pi$ are omitted. The different results for the Wilson coefficients are shown in Figure 2 as a function of the renormalization scale μ, setting $m_Q = 4.8\,\mathrm{GeV}$ corresponding to the b-quark mass. For the one-loop approximation we show a band obtained by varying the coupling between $\alpha_s(\mu)$ and $\alpha_s(m_Q)$. We note two important observations: (i) The one-loop and leading-logarithmic approximations give rather different results, which sometimes do not even have the same sign. This shows that the one-loop matching corrections at the scale $\mu = m_Q$ are important. (ii) For low renormalization scales $\mu \sim 1$–$2\,\mathrm{GeV}$, some of the coefficients, in particular $B_4(\mu)$ and $B_5(\mu)$, are numerically large despite the fact that they vanish at tree level. This shows that the effects of running are important, and thus a RG summation of the large logarithms $\alpha_s \ln(m_Q/\mu)$ becomes mandatory. The effects of matching and running can only be combined in a consistent, scheme-independent manner by going beyond the leading order. Our NLO results (shown by the solid lines) typically lie between the two approximations, except for $\mu \approx m_Q$, where they agree with the naive one-loop results.

6 Summary and an application

We have calculated at two-loop order the mixing of bilocal operators into local dimension-4 current operators containing a heavy and a light quark field. When combined with results obtained by previous authors, this completes the calculation of the 10×10 two-loop anomalous dimension matrix governing the renormalization-scale dependence of the operators appearing at order $1/m_Q$ in the heavy-quark expansion of the weak current $\bar{u}\gamma^\mu(1-\gamma_5)b$. We have presented expressions for the Wilson coefficients in this expansion at next-to-leading order in renormalization-group improved perturbation theory. We find that the next-to-leading corrections are numerically large and must be included for a meaningful determination of these coefficients. This makes our results relevant for phenomenological applications of the heavy-quark expansion.

The matrix elements of dimension-4 heavy–light current operators play an important role in heavy-flavor phenomenology. They appear, e.g., at order $1/m_Q$ in the heavy-quark expansion of meson decay constants [12], and of the semileptonic form factors describing the exclusive $\bar{B} \to \pi\, l\, \nu$ and $\bar{B} \to \rho\, l\, \nu$ transitions [13]. The primary motivation for our calculation, however, was the recent proposal for extracting the element $|V_{ub}|$ of the quark mixing matrix from the lepton invariant mass spectrum in inclusive $B \to X_u\, l\, \nu$ decays [14]. This spectrum can be obtained using a two-step operator product expansion (called "hybrid expansion") of a time-ordered product of two heavy–light currents [15]. Our results will help to reduce the perturbative uncertainty in this expansion, which will ultimately lead to an improved accuracy in the determination of $|V_{ub}|$.

At NLO in the hybrid expansion, one needs the expression for the combination of Wilson coefficients $(3B_4 + B_6)$ derived in the present work. We now present an exact result for this combination, evaluating the coefficients with $N = 3$ colors, but for an

arbitrary number of light quark flavors. We find

$$
\begin{aligned}
3B_4(\mu) + B_6(\mu) &= \frac{16}{9}\, x^{2/\beta_0} \left[1 + \frac{\alpha_s(m_Q)}{4\pi}\,(k_1 + k_2) - \frac{\alpha_s(\mu)}{4\pi} \left(k_1 - \frac{47}{6} + 11\Delta_{\rm RS} \right) \right] \\
&\quad - \frac{16}{9}\, x^{-1/\beta_0} \left[1 + \frac{\alpha_s(m_Q)}{4\pi}\, k_3 - \frac{\alpha_s(\mu)}{4\pi} \left(k_3 - k_2 - \frac{34}{3} - \Delta_{\rm RS} \right) \right] \\
&\quad + \frac{16}{\beta_0}\, x^{2/\beta_0} \ln x \left[1 + \frac{\alpha_s(m_Q)}{4\pi}\, k_4 - \frac{\alpha_s(\mu)}{4\pi} \left(k_4 + \frac{16}{3} + 2\Delta_{\rm RS} \right) \right] \\
&\quad - \frac{40}{9}\, \frac{\alpha_s(\mu)}{4\pi} \,,
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
k_1 &= -\frac{53}{6} + \left(\frac{1208}{9} + \frac{314\pi^2}{27} \right) \frac{1}{\beta_0} - \frac{1177}{\beta_0^2} \simeq 4.098 \,, \\
k_2 &= -\left(\frac{311}{6} + \frac{28\pi^2}{9} \right) \frac{1}{\beta_0 - 3} \simeq -15.476 \,, \\
k_3 &= \frac{49}{6} - \left(\frac{941}{18} + \frac{28\pi^2}{27} \right) \frac{1}{\beta_0} + \frac{107}{\beta_0^2} \simeq 2.206 \,, \\
k_4 &= -7 + \left(\frac{380}{9} - \frac{28\pi^2}{27} \right) \frac{1}{\beta_0} - \frac{214}{\beta_0^2} \simeq -6.243 \,,
\end{aligned}
\tag{41}
$$

and $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of the β-function. The numerical values in (41) refer to $n_f = 4$. In [16], we combine this result with an explicit calculation of the hadronic matrix elements in the hybrid expansion, finding that the scheme dependence parameterized by $\Delta_{\rm RS}$ disappears from the final result for the semileptonic $B \to X_u\, l\, \nu$ decay rate.

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